Written Exam Economics Summer 2021

Advanced Financial and Macro Econometrics

Comments on Solution

1 Co-integration and Pricing

[1] Both systems have $p = 3$.

Model (1.1) has $r = 2$ and therefore $p - r = 1$ stochastic trend. We can choose

$$\beta = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \\ 0 & \frac{3}{4} \end{pmatrix} \quad \text{and} \quad \beta_\perp = \begin{pmatrix} 1 \\ 4 \\ -\frac{16}{3} \end{pmatrix}.$$

Model (1.2) has $r = 1$ and therefore $p - r = 2$ stochastic trends. We can choose

$$b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad b_\perp = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

[2] Following the arguments in the lecture note, the solution should find the MA-solution for the stationary process $\beta' X_t$, i.e.

$$\varrho_t = \alpha \left(\beta'\alpha\right)^{-1} \left(\sum_{i=0}^{t-1} \left(I_r + \beta'\alpha\right)^i \beta'\epsilon_{t-i} + \left(I_r + \beta'\alpha\right)^t \beta' X_0 \right).$$

Similarly for S_t.

[3] The two systems have three co-integrating relationships. Following the lecture note, the solution should argue that they are candidates for trading pairs. Whenever the deviation from equilibrium is non-zero (or significantly non-zero) the agent could buy the underpriced stock and sell the overpriced stock and wait for reversal to equilibrium. More details could be given.

[4] Now a system of $p = 4$ variables, $Z_t = (x_{1,t}, x_{3,t}, y_{1,t}, y_{3,t})'$, is considered.

[4.1] In this case $r = 2$. We can choose

$$\beta^* = \begin{pmatrix} 1 & 0 \\ \frac{3}{16} & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \beta^*_\perp = \begin{pmatrix} 1 & 0 \\ -\frac{16}{3} & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

[4.2] Next, we are informed that $x_{1,t} - y_{3t}$ is also stationary. We can choose

$$\beta^* = \begin{pmatrix} 1 & 0 & 1 \\ \frac{3}{16} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \beta_\perp^* = \begin{pmatrix} 1 \\ -\frac{16}{3} \\ 1 \\ 1 \end{pmatrix}.$$

[5] Let $y_{3,t}$ denote the market portfolio and consider the system for $Y_t = (y_{1,t}, y_{2,t}, y_{3,t})'$:

$$\begin{pmatrix} \Delta y_{1,t} \\ \Delta y_{2,t} \\ \Delta y_{3,t} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}' \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{pmatrix} + \begin{pmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{pmatrix}, \quad \text{and} \quad O = \begin{pmatrix} O_{11} & O_{12} & O_{13} \\ O_{21} & O_{22} & O_{23} \\ O_{31} & O_{32} & O_{33} \end{pmatrix}.$$

[5.1] The short-run conditional pricing beta for the first asset is

$$B_t^1 = \frac{\text{cov}_{t-1}(\Delta y_{1,t}, \Delta y_{3,t})}{V_{t-1}(\Delta y_{3,t})} = \frac{\text{cov}_{t-1}(e_{1,t}, e_{3,t})}{V_{t-1}(e_{3,t})} = \frac{O_{13}}{O_{33}}.$$

In this case the conditional variance is constant and B_t^1 is not time-varying. By conditioning in the Gaussian distribution, the model $y_{1,t}, y_{2,t} \mid y_{3,t}$ is

$$\begin{pmatrix} \Delta y_{1,t} \\ \Delta y_{2,t} \end{pmatrix} = \omega \Delta y_{3,t} + \begin{pmatrix} a_1 - \omega a_3 \\ a_2 - \omega a_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}' \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{pmatrix} + \begin{pmatrix} e_{1,t}^* \\ e_{2,t}^* \end{pmatrix},$$

where $\omega = O_{13}/O_{33} = B_t^1$ can be easily estimated.
The OLS estimator in the linear regression

$$\Delta y_{1,t} = \lambda \Delta y_{3,t} + \nu_t,$$

is given by

$$\hat{\lambda} = \frac{\frac{1}{T}\sum_{t=1}^T \Delta y_{1,t} \Delta y_{3,t}}{\frac{1}{T}\sum_{t=1}^T (\Delta y_{3,t})^2} = \frac{\frac{1}{T}\sum_{t=1}^T (a_1 b' Y_{t-1} + e_{1,t})(a_3 b' Y_{t-1} + e_{3,t})}{\frac{1}{T}\sum_{t=1}^T (a_3 b' Y_{t-1} + e_{3,t})^2}.$$

The probability limit is given by

$$\hat{\lambda} \to_P \lambda = \frac{E(a_1 b' Y_{t-1} + e_{1,t})(a_3 b' Y_{t-1} + e_{3,t})}{E(a_3 b' Y_{t-1} + e_{3,t})^2} = \frac{a_1 a_3 \Sigma_{bb} + O_{13}}{a_3^2 \Sigma_{bb} + O_{33}},$$

where Σ_{bb} is the variance of $b' Y_t$. For $\hat{\lambda}$ to be a consistent estimator of $B_{1,t}$, the requirement is that $a_3 = 0$, such that the omitted variable, $b' Y_{t-1}$, is uncorrelated with the regressor, $\Delta y_{3,t}$.

[5.2] The solution should explain that according to the CAPM model the pricing beta, B_t^1, is the relevant measure of risk in a well-diversified portfolio. The investor is compensated for this systematic risk, while the idiosyncratic risk can be diversified away and is therefore not priced according to the CAPM.

[5.3] Because it is market neutral, a pairs-trading portfolio has a zero beta, and in terms of CAPM it should therefore pay the risk-free rate. One main difference between CAPM and the pairs-trading strategy is that CAPM is an equilibrium pricing theory, while pairs-trading strategies tries to identify prices out of equilibrium.

[6] In the extended case

$$\begin{aligned} O_t &= \begin{pmatrix} O_{11,t} & O_{12,t} & O_{13,t} \\ O_{21,t} & O_{22,t} & O_{23,t} \\ O_{31,t} & O_{32,t} & O_{33,t} \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{1,t} & 0 & 0 \\ 0 & \sigma_{2,t} & 0 \\ 0 & 0 & \sigma_{3,t} \end{pmatrix} \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} \begin{pmatrix} \sigma_{1,t} & 0 & 0 \\ 0 & \sigma_{2,t} & 0 \\ 0 & 0 & \sigma_{3,t} \end{pmatrix} \\ &= \begin{pmatrix} \sigma^2_{1,t} & \rho_{12}\sigma_{1,t}\sigma_{2,t} & \rho_{13}\sigma_{1,t}\sigma_{3,t} \\ \rho_{12}\sigma_{1,t}\sigma_{2,t} & \sigma^2_{2,t} & \rho_{23}\sigma_{2,t}\sigma_{3,t} \\ \rho_{13}\sigma_{1,t}\sigma_{3,t} & \rho_{23}\sigma_{2,t}\sigma_{3,t} & \sigma^2_{3,t} \end{pmatrix}. \end{aligned}$$

Here the conditional pricing beta is

$$B_t^1 = \frac{O_{13,t}}{O_{33,t}} = \rho_{13}\frac{\sigma_{1,t}}{\sigma_{3,t}},$$

which is time varying.

[7] The estimates of the model are

```
Robust Standard Errors (Sandwich formula)
                      Coefficient  Std.Error  t-value  t-prob
Part: Dy1
Cst(M)                   0.017293   0.051552   0.3354  0.7375
lagspread (M)           -0.157069   0.025285   -6.212  0.0000
Cst(V)                   0.928276   0.082704    11.22  0.0000
ARCH(Alpha1)             0.297931   0.060889    4.893  0.0000

Part: Dy2
Cst(M)                  -0.044709   0.075182  -0.5947  0.5524
lagspread (M)            0.020588   0.032302   0.6374  0.5243
Cst(V)                   2.022310    0.15978    12.66  0.0000
ARCH(Alpha1)             0.322421   0.068444    4.711  0.0000

Part: Dy3
Cst(M)                  -0.020567   0.088355  -0.2328  0.8161
lagspread (M)            0.014188   0.039903   0.3556  0.7224
Cst(V)                   2.376187    0.22266    10.67  0.0000
ARCH(Alpha1)             0.365069   0.067735    5.390  0.0000

Part: Correlation
rho_21                   0.791882   0.018362    43.13  0.0000
rho_31                   0.760147   0.020258    37.52  0.0000
rho_32                   0.790171   0.016995    46.49  0.0000

No. Observations :          399  No. Parameters   :        15
No. Series       :            3  Log Likelihood   : -1767.467
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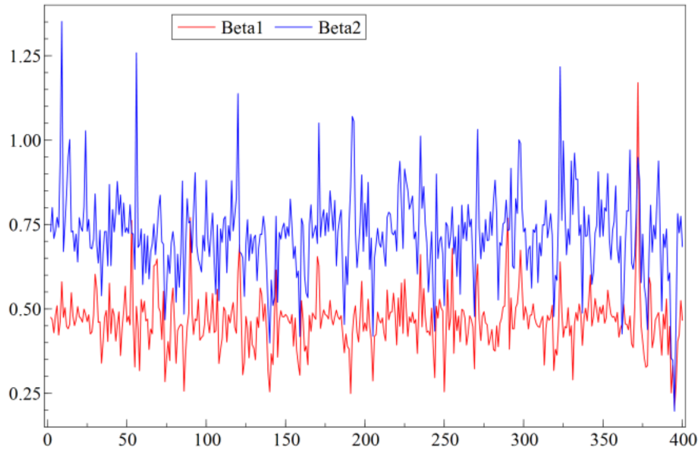
The solution should also report estimated conditional variances, covariances and correlations, noting that the latter are constant by construction.

The estimated time-varying betas are gives as



The simple multivariate ARCH allows estimation of the time varying betas. The solution could discuss that a model with constant conditional correlations and ARCH(1) specifications for the conditional variances may be too simple to model time-varying betas. With $B_t^1 = \rho_{13}\sigma_{1,t}/\sigma_{3,t}$, all time variation comes from the relative conditional standard deviations, and they will typically not show very persistent movements. As a result, fluctuations in betas implied by the model are rather transitory.

2 Multivariate GARCH-X

[8] Drift criterion:

[8.1] It holds, that

$$\begin{aligned}
E\left(1+v_t'v_t|v_{t-1}=v\right) &= 1+E\left(Y_t'Y_t+x_t^2|v_{t-1}=v\right)\\
&= 1+\operatorname{tr}\left(\Omega_t\right)+\sigma_x^2\\
&= \left(\omega_1^2+\omega_2^2+\alpha_2 y_2^2+\alpha_1 y_1^2+2\gamma x^2\right)+\sigma_x^2\\
&= c+\alpha_2 y_2^2+\alpha_1 y_1^2+2\gamma x^2\\
&= c+v'Av\leq c+\max\left(\alpha_1,\alpha_2,2\gamma\right)v'v.
\end{aligned}$$

[8.2] Hence $\max\left(\alpha_1,\alpha_2,2\gamma\right)<1$ is a sufficient condition.

[8.3] As $Y_t'Y_t+x_t^2=y_{1t}^2+y_{2t}^2+x_t^2$ the covariance does not change this.

[9] Testing:

[9.1] It follows that

$$\ell_T\left(\theta\right)=-\frac{1}{2}\sum_{t=1}^{T}\left(\log\det\left(\Omega_t\right)+\operatorname{tr}\left\{Y_tY_t'\Omega_t^{-1}\right\}\right)$$

with $\partial \ell_T(\theta)/\partial\gamma = \frac{1}{2}\sum_{t=1}^{T} \frac{x_{t-1}^2}{\sigma_{1t}^2}\left[y_{1t}^2/\sigma_{1t}^2 - 1\right] + \frac{x_{t-1}^2}{\sigma_{2t}^2}\left[y_{1t}^2/\sigma_{2t}^2 - 1\right]$, as

$$\begin{aligned} \partial\left[\log\det(\Omega_t)\right]/\partial\gamma &= \partial\left[\log\det\left(D_t^2\right)\right]/\partial\gamma = \partial\left[\log\sigma_{1t}^2 + \log\sigma_{2t}^2\right]/\partial\gamma \\ &= \left[1/\sigma_{1t}^2 + 1/\sigma_{2t}^2\right]x_{t-1}^2 \end{aligned}$$

$$\begin{aligned} \partial\left[\operatorname{tr}\left\{Y_tY_t'\Omega_t^{-1}\right\}\right]/\partial\gamma &= \partial\left[y_{1t}^2/\sigma_{1t}^2 + y_{2t}^2/\sigma_{2t}^2\right]/\partial\gamma \\ &= -\left[y_{1t}^2/\sigma_{1t}^4 + y_{2t}^2/\sigma_{2t}^4\right]x_{t-1}^2 \end{aligned}$$

Hence

$$\partial\ell_T(\theta)/\partial\gamma|_{\theta=\theta_0} = \frac{1}{2}\sum_{t=1}^{T}\frac{x_{t-1}^2}{\sigma_{1t}^2}\left[z_{1t}^2 - 1\right] + \frac{x_{t-1}^2}{\sigma_{2t}^2}\left[z_{2t}^2 - 1\right]$$

With $\xi_t = x_{t-1}^2\left(\frac{1}{\sigma_{1t}^2}, \frac{1}{\sigma_{2t}^2}\right)\operatorname{vec}(z_tz_t' - I_2)$, it follows that ξ_t is a MGD wrt. $\mathcal{F}_{t-1} = \{(x_{t-k}, Y_{t-k}),\ k \geq 1\}$ and

$$E\left(\left\{\frac{x_{t-1}^2}{\sigma_{1t}^2}\left[z_{1t}^2 - 1\right] + \frac{x_{t-1}^2}{\sigma_{2t}^2}\left[z_{2t}^2 - 1\right]\right\}^2 |\mathcal{F}_{t-1}\right) = 2\frac{x_{t-1}^4}{\sigma_{1t}^4} + 2\frac{x_{t-1}^4}{\sigma_{2t}^4}.$$

And, moreover

$$T^{-1}\sum_{t=1}^{T}\left(\frac{x_{t-1}^4}{\sigma_{1t}^4} + 2\frac{x_{t-1}^4}{\sigma_{2t}^4}\right) \to_P E\left(\frac{x_{t-1}^4}{\sigma_{1t}^4} + 2\frac{x_{t-1}^4}{\sigma_{2t}^4}\right) = m$$

where $m < \infty$ if $\gamma_0 > 0$ since $\left(x_{t-1}^2/\sigma_{it}^2\right)^2 \leq 1/\gamma_0^2$. No moment restrictions needed. Hence just stationarity and ergodicity is sufficient.

[9.2] If the additional "usual" (to be included) reg. conditions hold on information and third derivatives - this implies that $\hat{\theta}$ can be reported with standard errors. However, this does not include the LR test for $\gamma = 0$ (and t-statistics).

[9.3] When $\gamma_0 = 0$, we need $E\left(x_{t-1}^4\right) < \infty$. This is not a further requirement as under Ass. eXo x_t is assumed Gaussian.

[9.4] The LR($\gamma = 0$) is asymptotically "$\frac{1}{2}\chi_1^2$" distributed. Mention e.g.: (a) $\alpha_{i0} > 0$, (b) well-known implications (e.g. 5% quantile is the 10% quantile of the χ_1^2)

[10] Bootstrap:

[10.1] With $\hat{z}_t = \hat{\Omega}_t^{-1/2}Y_t$ the estimated residuals, set $\hat{z}_t^c = \hat{z}_t - T^{-1}\sum_{t=1}^{T}\hat{z}_t$ and empiricalV$\left(\hat{z}_t^c\right) = T^{-1}\sum_{t=1}^{T}\hat{z}_t^c\hat{z}_t^{c\prime}$. Define the standardized residuals (explain why this is needed)

$$\hat{z}_t^s = \left[\text{empiricalV}\left(\hat{z}_t^c\right)\right]^{-1/2}\hat{z}_t^c.$$

A bootstrap-sample $\{Y_t^*\}$ can then be constructed as e.g.

$$Y_t^* = \left(\Omega_t^*\right)^{1/2} z_t^*$$

where z_t^* are sampled from $\hat{z}_t^s$ (wild, with replacement etc.), and

$$\Omega_t^* = D_t^* \Gamma^* D_t^*.$$

Here, with $\tilde{\theta}$ denoting the restricted estimator (explain why),

$$\Gamma^* = \begin{pmatrix} 1 & \tilde{\rho} \\ \tilde{\rho} & 1 \end{pmatrix} \quad \text{and} \quad D_t^* = \text{diag}\,(\sigma_{it}^*)_{i=1,2} = \begin{pmatrix} \sigma_{1t}^* & 0 \\ 0 & \sigma_{2t}^* \end{pmatrix},$$

with

$$\sigma_{it}^{*2} = \tilde{\omega}_i + \tilde{\alpha}_i y_{it-1}^2$$

The LR^* statistic for $\gamma = 0$ is then computed. And as usual, this is repeated say $B = 399$ times - more text can be added.

[10.2] A main issue is if α_{10} and α_{20} equal zero or not (when $\gamma_0 = 0$) as this will affect the limiting distribution.